p27. 6. Exercises Solutions

1. The cube of any even number is divisible by 8.
Let $2k$ be any even number, for some integer k
Then $(2k)^3 = 8k^3$ which is a multiple of 8 since k^3 must also be an integer
so any even number cubed is divisible by 8.

2. The sum of the first n odd numbers is n^2
Let's just think about two with numbers:
1st 3 odd nos: 1, 3, 5 , their sum is $1+3+5=9$ which is $n^2=3^2=9$
Now let's consider the sum of the 1st n odd numbers:
$S_n = 1+3+5+\ldots+(2n-5)+(2n-3)+(2n-1)$
Writing this in reverse we can also say that
$S_n = (2n-1)+(2n-3)+(2n-5)+\ldots+5+3+1$
Adding these together we get
$2S_n = [1+(2n-1)]+[3+(2n-3)]+[5+(2n-5)]+\ldots+[(2n-5)+5]+[(2n-3)+3]+[(2n-1)+1]$
$\therefore 2S_n = 2n + 2n + 2n + \ldots + 2n + 2n + 2n$
i.e. $2S_n = n \times 2n$
$2S_n = 2n^2$
$\therefore S_n = n^2$
$\Rightarrow S_n = 1+3+5+\ldots+(2n-1) = n^2$ so the sum of the 1st n odd nos. is n^2.

3. $\cos\theta\cot\theta + \sin\theta \equiv \operatorname{cosec}\theta$
Starting with the LHS: $\cos\theta\cot\theta+\sin\theta \equiv \cos\theta \times \frac{\cos\theta}{\sin\theta} + \sin\theta \equiv \frac{\cos^2\theta}{\sin\theta} + \frac{\sin^2\theta}{\sin\theta}$

$\equiv \frac{\cos^2\theta+\sin^2\theta}{\sin\theta} \equiv \frac{1}{\sin\theta} \equiv \operatorname{cosec}\theta = \text{RHS}$

It is undefined when $\sin\theta=0$, i.e when $\theta = 0, \pi, 2\pi \ldots$ so when $\theta = \pm n\pi$ $\forall n \in \mathbb{Z}$

4. Let's call our four consecutive integers $n, n+1, n+2$ and $n+3$
Then $(n+2)(n+3) - n(n+1) = n^2+5n+6 - (n^2+n) = 4n+6$
The sum $= n+n+1+n+2+n+3 = 4n+6$
So the difference between the product of the last 2 and the product of the first 2 is equal to the sum of the four consecutive integers.

5. To show the graphs are identical we need to show that $y = \sin x \cos x + 1$ is always above the horizontal axis.
Both $\sin x$ and $\cos x$ have a max value of 1 and min value of -1
So the lowest possible value of $\sin x \cos x$ is $1 \times -1 = -1$
So for all values of x, the lowest possible value of $\sin x \cos x + 1$ is $-1+1=0$
So the graph of $y = \sin x \cos x + 1$ does not go below the horizontal axis so it will be identical to $y = |\sin x \cos x + 1|$

6. Let 4 consecutive numbers be $n, n+1, n+2, n+3$
Their product is $n(n+1)(n+2)(n+3) = (n^2+n)(n^2+5n+6) = n^4+5n^3+6n^2+n^3+5n^2+6n$
$= n^4+6n^3+11n^2+6n$
We want to show that $n^4+6n^3+11n^2+6n+1$ is a perfect square
i.e of the form $(n^2+an+b)^2 = n^4+2an^3+(a^2+2b)n^2+(2ab)n+b^2$
Comparing coefficients: $2a = 6 \Rightarrow a = 3$
$a^2+2b = 11 \Rightarrow 9+2b = 11 \Rightarrow b = 1$
So we can write $n(n+1)(n+2)(n+3) = (n^2+3n+1)^2 - 1$
i.e. the product of four consecutive numbers is one less than a perfect square

7. $9^n - 1 = 3^{2n} - 1 = (3^n-1)(3^n+1)$
3^n is odd for all n, so 3^n-1 and 3^n+1 are both even.
In fact $3^n-1+2 = 3^n+1$ so 3^n-1 and 3^n+1 are consecutive even numbers
In any pair of consecutive even numbers, one must be a multiple of 4 and the other one is even so is a multiple of 2.
Hence their product, and the product of $(3^n-1)(3^n+1) = 9^n-1$ must be a multiple of 8.

8. $p^2-1 = (p-1)(p+1)$
$p-1, p$ and $p+1$ are 3 consecutive integers so one is a multiple of 3
p is a prime >3 so either $p-1$ or $p+1$ is a multiple of 3 so p^2-1 is a multiple of 3
p is a prime >3 so must be odd, which means $p-1$ and $p+1$ are consecutive even numbers. By the same reasoning as in Q7, p^2-1 must thus be a multiple of 8. Since p^2-1 is a multiple of 8 and a multiple of 3, it must be a multiple of $3 \times 8 = 24$.