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Modes of reasoning in Israeli seventh grade mathematics textbook explanations

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Goal

To examine justifications and explanations in Israeli 7th grade mathematics textbooks

Our focus:

- The modes of reasoning offered
- Justifications in algebra and in geometry
- Textbooks of limited- vs. regular/extended scope

Sample: statements

- Eight key mathematical statements
- Selecting from the Israeli national curriculum

Sample: algebra statements

- The distributive property – for every three numbers a,b,c: a(b+c)=ab+ac.
- The product of two negative numbers is a positive number.
- Division by zero is undefined.
- Performing a basic operation on both sides of an equation maintains their balance.

Sample: geometry statements

- The area formula for a trapezium with bases a,b and altitude h is $(a+b)h/2$.
- The area formula for a circle with radius r is πr^2.
- Angle sum of a triangle is 180^{o}
- The corresponding angles between parallel lines are equal.

Sample: textbooks

Eight 7th grade textbooks:

- Six (A-F) are of regular/extended scope
- Two (G-H) are of limited scope

Textbook sections introducing each statement:

- Explanatory texts
- related tasks and problems

Total: 549 pages (48-94 pages from each textbook)

Methods of analysis

- Identifying distinct justifications in each textbook section
- Classifying each justification for its mode of reasoning (following Stacey & Vincent, 2009)
- Comparing relevant frequencies:
  - The modes of reasoning offered
  - Justifications in algebra and geometry
  - Textbooks of limited- vs. regular/extended scope

Modes of reasoning framework

(Stacey & Vincent, 2009)

In Australian 8th grade textbook explanations:

- Appeal to authority
- Qualitative analogy
- Experimental Demonstration
- Concordance of a rule with a model
- Deduction using a model
- Deduction using a specific case
- Deduction using a general case

Findings: modes of reasoning offered

In Israeli 7th grade textbook explanations:

- Appeal to authority
- Qualitative analogy
- Experimental Demonstration
- ~~Concordance of a rule with a model~~
- Deduction using a model
- Deduction using a specific case
- Deduction using a general case

Appeal to authority

Reliance on an external source of authority / Null-explanation.

E.g. The area formula for a circle:

Reminder: The **perimeter** of a circle with radius r is $2 \cdot \pi \cdot r$.
The **area** of a disc with radius r is $\pi \cdot r^2$.

Qualitative analogy

Reliance on a surface similarity to non-mathematical situations.

E.g., Multiplication of signed numbers

?! 15 **Wordplay**: match each statement with the corresponding sign rule.

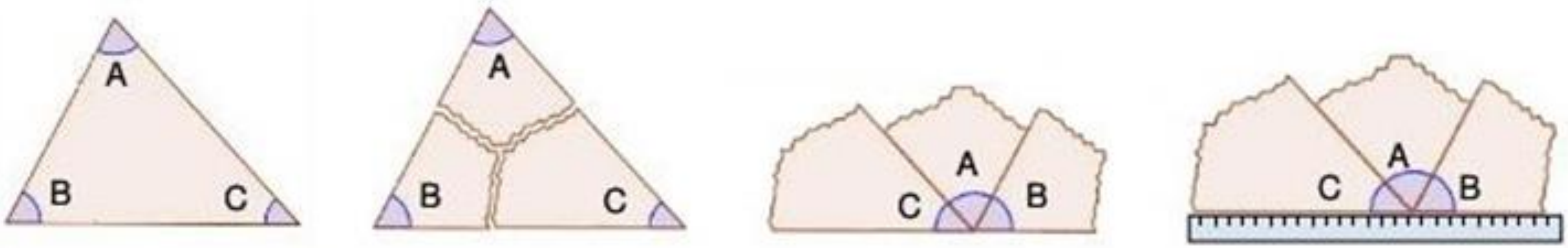
Discussion

- The friend of my friend is my friend.
- The friend of my enemy is my enemy.
- The enemy of my friend is my enemy.
- The enemy of my enemy is my friend.

Experimental demonstration

Identifying a pattern after checking selected examples.

E.g., The angle sum of a triangle is 180°



1. Cut any triangle out of a sheet of paper
2. Tear the triangle to pieces, such that each piece contains one angle of the triangle.
3. Place the three angles of the triangle adjacently. What did you get?
4. It appears that the three angles together form a straight angle. Test that with a ruler or a rectangular sheet.

Deduction using a model

A model that serves to illustrate a mathematical structure.

E.g., Balancing equations

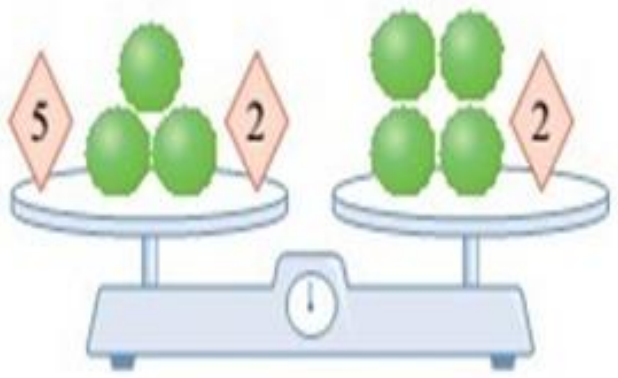
The illustrated scales are balanced. The number in each rhombus represents units.

a. Which operations maintain the balance?

1 Subtracting 2 units from each side.

2 Adding 6 units to each side.

3 Moving 2 units from the right side to the left.

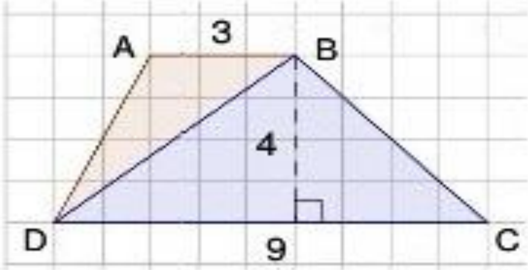


Deduction using a specific case

An inference process conducted using a special case.

E.g., The area formula of a trapezium

Draw the diagonal BD and split the area of the trapezium to two triangles: ΔABD and ΔDBC. The two triangles share an altitude (4 cm). The area of the trapezium ABCD is the sum of the areas of the two triangles:



$$S_{\Delta ABD} = \frac{3 \cdot 4}{2} \qquad S_{\Delta DBC} = \frac{9 \cdot 4}{2}$$

$$S_{ABCD} = S_{\Delta ADB} + S_{\Delta DBC} = \frac{3 \cdot 4 + 9 \cdot 4}{2} = \frac{(3+9) \cdot 4}{2} = 24\ \text{cm}^2$$

Let's observe the last step and identify the components of the formula.

We see that we can find the area of a trapezium by finding the sum of lengths of the bases, times the length of the altitude, and dividing the product by 2.

short base, long base, altitude

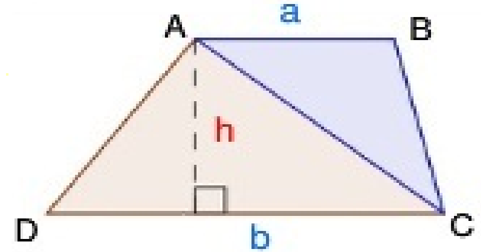
$$S_{ABCD} = \frac{(3+9) \cdot 4}{2}$$

Deduction using a general case

An inference process conducted using a general case.

E.g., The area formula of a trapezium

Let's show that the area formula of a trapezium is true in all cases.
Fill in the missing steps in the proof.



- Draw the diagonal AC in the trapezium. It divides the trapezium ABCD into two triangles ΔABC and ΔADC.
- Therefore: $S_{ABCD} = S_{\Delta ABC} + S_{\Delta ADC}$
- The triangles ΔABC and ΔADC share an altitude h (why?)
- Let's write the area formula. Fill in the missing steps:

$$S_{ABCD} = S_{\Delta ABC} + S_{\Delta ADC} = \frac{a \cdot h}{2} + \frac{\quad}{2} = \frac{ah + \quad}{2} = \frac{(a+b) \cdot h}{2}$$

In other words: **The area of a trapezium is the sum of the lengths of the bases, times the length of the altitude, divided by 2.**

Statement \ Textbook	A	B	C	D	E	F	G	H
Distributive law	m,m	e,m	m,s	m,m ,s	e,m	m,m,s,g	e,m	m,s
Multiplication of negative integers	q,m,m, m,s,g	s,s	s,g	s,g	s,g	g	a,e	g
Division by zero	m,s,s	s	s,g	s,g	s,g	s,g	s	s,g
Balancing equations	e,m	s	e,m,s	s	e,m,g	m,s	e	m,s
Area of a trapezium	s,s,s,g	s,s, g,g,g	e,s,s, s,g	e,e,e, s,s	e,e,e,s	e,e,s,g	e,e,e ,g	e,e,e, s
Area of a disc	g	g	g	g	a	e,g	g	g
Angle sum of a triangle	e,g,g	e,e,e, g,g	e,e,g	e,g,g	e,e,g	e,e,e,g, g,g,g	e,e,g	e,e,e
Corresponding angles	e,s,g	e,g	e,e	e,g	e	e	e	e,e

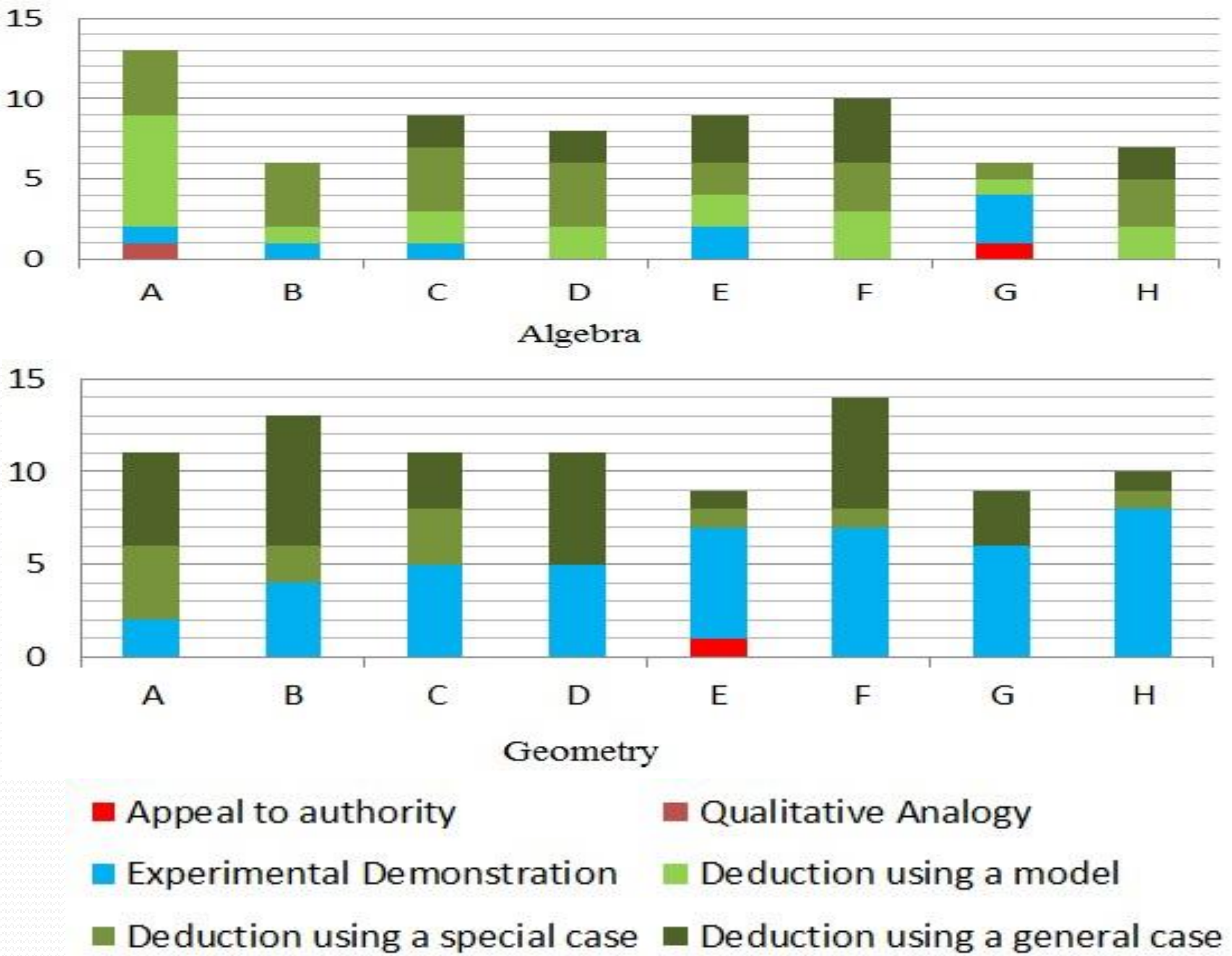
Legend: a= appeal to authority; q= qualitative analogy; e= experimental demonstration; m= deduction using a model; s= deduction using a specific case; g= deduction using a general case.

Modes of reasoning offered

- Commonly, several modes of reasoning are used
- Almost all explanations are of deductive or empirical modes

(cf. In Australian textbooks for similar topics: 17% neither deductive nor empirical (Stacey & Vincent, 2009))

Findings: algebra and geometry

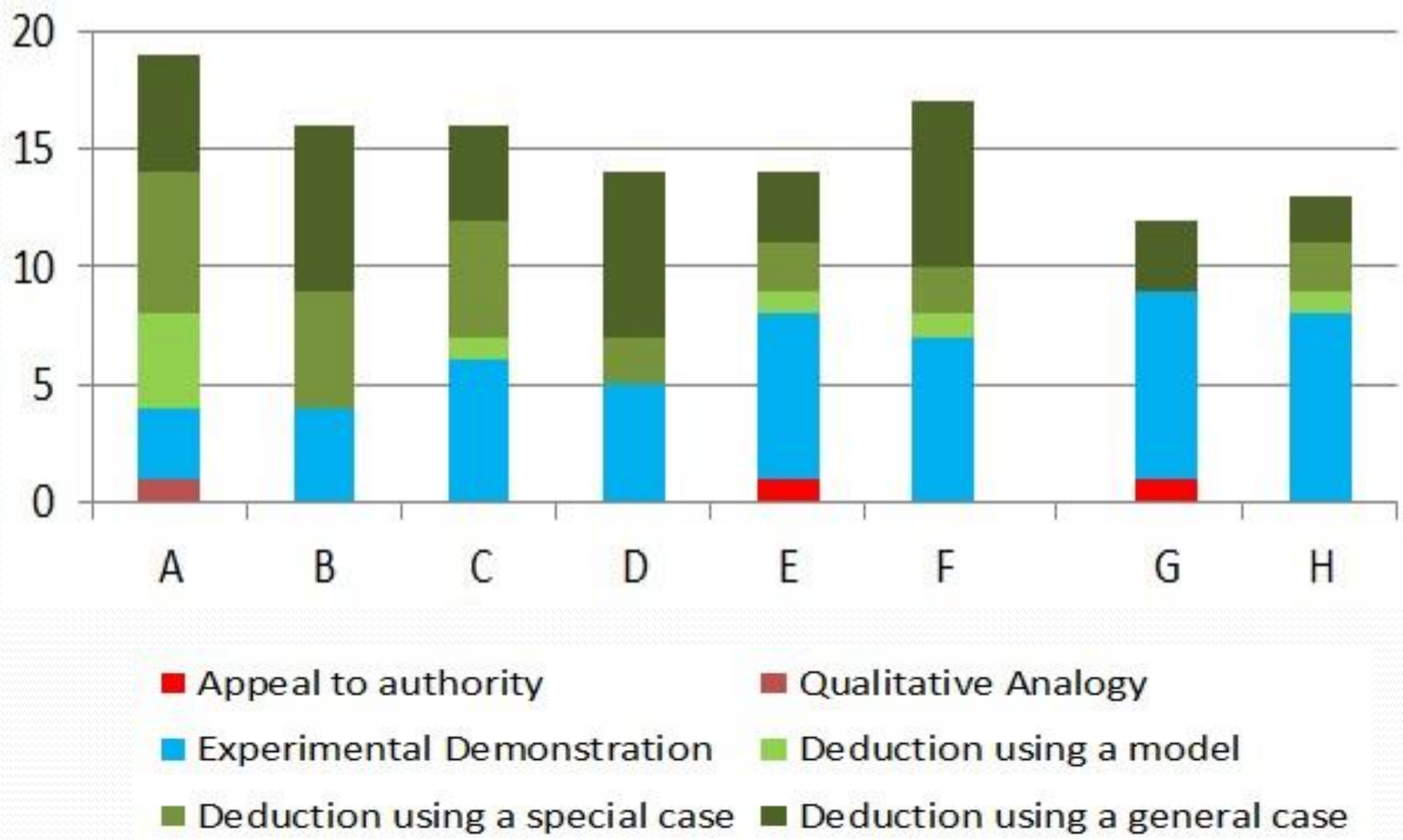


Algebra and geometry

- Algebra: statements typically justified by deductive modes of reasoning
- Geometry: usually justified both by deductive modes of reasoning and by an empirical mode of reasoning

- Initially surprising, given the historic bias
- *Deduction using a general case* is more common in geometry than in algebra

Findings: target student population



Target student population

- Empirical justifications:
  a greater percentage in textbooks of limited scope
- Deductive justifications:
  a greater percentage in textbooks of regular / extended scope

- Influences the opportunities of students with low achievements to learn how to justify in mathematics

Thank you

Concordance of a rule with a model

Comparing specific results of a model and of a rule:

E.g., Division of a fractions